Some Key Junctures in Relational Thinking

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This study uses number sentences involving one and two unknown numbers to identify some key junctures between relational thinking on number sentences and an ability to deal with sentences involving literal symbols. Number sentences involving two unknown numbers distinguish between students who are restricted to computational approaches and those who can genuinely engage in relational thinking. Furthermore, such sentences allow the identification of different stages of relational thinking.

Rationale for the Study

Irwin and Britt (2005) argue that the methods of compensating and equivalence that some students use in solving number sentences may provide a foundation for algebraic thinking (p. 169). Carpenter and Franke (2001) refer to the thinking underpinning this kind of strategy as relational thinking. There is, however, still a debate about whether relational thinking when applied to number sentences can be properly described as algebraic. Here, the view of Jacobs, Franke, Carpenter, Levi and Battey (2007) seems very appropriate:

> One could debate whether our characterization of relational thinking in arithmetic represents a way of thinking about arithmetic that provides a foundation for learning algebra or is itself a form of algebraic reasoning. A case could be made either way. One fundamental goal of integrating relational thinking into the elementary curriculum is to facilitate students' transition to the formal study of algebra in the later grades so that no distinct boundary exists between arithmetic and algebra. (p. 261)

A clearer picture is needed of "how these ideas … develop in children's thinking, and the critical junctures in this development" (Katz, 2007, p. 10). Understanding these critical junctures is important in identifying the development of relational thinking. For example, if children are able to apply successfully ideas of equivalence and compensation to solve number sentences involving all four arithmetical operations, can they also use ideas of equivalence and compensation to deal successfully with sentences involving literal symbols? Moreover, is it possible to identify in students' justifications of their relational thinking any linguistic markers of development of their algebraic thinking?

Methodology

Design of questionnaire

Previous research (Stephens, 2007) used missing number sentences involving only addition and subtraction. Given the central role of the ideas of equivalence and compensation in relational thinking, it is important to get a picture of students' relational thinking across the four operations as they near the end of primary school or begin secondary school. In this study involving students in Year 6 and Year 7, number sentences involving all four arithmetical operations were therefore included. These took the form of sentences with one missing number where the value of that number can be found either by relational thinking or by computation. Examples of items used in the study were as follows:

$43 + \square = 48 + 76$, $39 - 15 = 41 - \square$, $\square \times 5 = 20 \times 15$, $21 \div 56 = \square \div 8$

Students were also asked to write briefly how they found the value of the missing number.

Previous studies (Stephens, Isoda, & Inprashita, 2007; Stephens, 2007) identified a small group of students who successfully used computational methods to solve missing number sentences of this type, but who were also able to deal, more or less successfully, with expressions involving literal symbols. This group of students appeared to opt for computational methods to deal with missing number sentences, but were quite capable of using ideas of equivalence and compensation to solve sentences involving literal symbols; and were clearly different from those students who could use only computational methods.

The study therefore needed to include a type of numbers sentence where students are "pushed" to think relationally. Number sentences involving two unknown numbers, such as 18 + (Box A) = 20 + (Box B), seem to have this potential. While it is possible to use computational methods to give particular instances of correct sentences taking this form, identifying a general structural relationship requires students to move beyond computational thinking. For example, a clear relational explanation might say that the above sentence will always be true *as long as the number in Box A is two more than the number in Box B*. Being able to derive a correct mathematical generalisation from numerical examples is key element of algebraic reasoning (Carpenter and Franke, 2001; Zazkis and Liljedahl, 2002).

Number sentences involving two unknown numbers – across all four operations – were included in the study. These allowed some scope for computational approaches, but, following Fujii (2003), identifying the critical numbers and the relational elements embodied in these expressions required that students move beyond computation and focus especially on expressing the underlying mathematical structure.

Finally, to determine if, as Jacobs et al. (2007) suggest, equivalence and compensation provide a foundation for algebraic thinking, some questions involving literal symbols were included. Therefore, several questions modelled after the research programme, *Concepts in Secondary Mathematics and Science* (CSMS, see Hart, 1981), asked students: What can you say about c and d in the following mathematical sentence? $c + 2 = d + 10$

This third type of question allowed students to say that this sentence will be true for *any* values of c and d provided c is 8 more than d. But other students may fall short of this simply giving several values of c and d for which the sentence is true. Other students may say "c is more than d" but cannot specify the relationship. The value of questions such as these is that they can be given partial or complete relational interpretations.

A questionnaire, consisting of eight pages, was comprised of these three types of questions with each type distributed across the four operations. A first type comprised missing number sentences involving one unknown number. A second type comprised arithmetical sentence with two unknown numbers. A third type was structurally similar to the second type but explicitly included literal symbols.

Type 1: Missing Number Sentences with One Unknown Number

The position of the box denoting a missing number was varied for each item. The given numbers were chosen so as to provide numbers either side of the equal sign such that a relational approach to finding the missing number was attractive. Of course, it was also possible for students to solve each question by computation. After each item, space was provided for students to write how they had found the missing number. Similar Type 1 questions (Questions 3, 5, and 7) were used for subtraction, multiplication and division, with following sample items: $104 - 45 = \square - 46$; $36 \times 25 = 9 \times \square$; and $18 \div \square = 6 \div 10$. Figure 1 shows Type 1 questions used for addition used on the first page:

1. For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

$23 + 15 = 26 + \square$	$73 + 49 = \square + 47$
$43 + \square = 48 + 76$	$\square + 17 = 15 + 24$

Figure 1. Question 1 involving addition and Type 1 questions.

Types 2 and 3: Sentences Involving Two Related But Unknown Numbers

Each even-numbered page opened with a Type 2 question using two boxes, denoted by Box A and Box B, and employing one arithmetical operation. Type 2 questions are exemplified in parts (a) to (d) in Figure 2. These were then followed by a related Type 3 question, shown in part (e), involving the same arithmetical operation and literal symbols.

2. Can you think about the following mathematical sentence:

18 + ☐ = 20 + ☐

Box A Box B

(a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

18 + ☐ = 20 + ☐

Box A Box B

18 + ☐ = 20 + ☐

Box A Box B

18 + ☐ = 20 + ☐

Box A Box B

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

(e) What can you say about c and d in this mathematical sentence? $c + 2 = d + 10$

Figure 2. Question 2 involving addition and Type 2 and Type 3 questions.

This same format was used for questions involving the other three operations. Question 4 for subtraction was: 72 – (Box A) = 75 – (Box B). Question 6 for multiplication was: 5 × (Box A) = 10 × (Box B). Question 8 for division was: 3 ÷ (Box A) = 15 ÷ (Box B). Corresponding Type 3 items were: $c - 7 = d - 10$; $c \times 2 = d \times 14$; and $c \div 8 = d \div 24$.

The Sample

The sample was drawn from Year 6 and Year 7 students in two schools, one in Australia and one in China. The Chinese sample consisted of two intact classes consisting of 32 students in Year 6 and 36 students in Year 7. In the Australian school, one Year 6 class of 25 students was involved and three Year 7 classes consisting of 71 students altogether. The sample was a convenience sample. The performances of students are therefore not presented as being normative of schools in each country, and may reflect the teaching they have received. It is, however, possible to examine students' performances on the three types of sentences, and to track what students do over certain junctures. Translation of the questionnaire into Chinese was prepared by faculty members at an Eastern Chinese university. Graduate students at the same university and two Chinese speaking graduates in Australia assisted with the translation of students' responses. Each student's written responses were read independently by two markers. A very high degree of consistency of classification was evident across markers in both countries.

Key Questions to Be Investigated

Several questions guided an analysis of students' responses. First, what evidence was there of computational and relational approaches to Type 1 sentences and what forms did this thinking take? Then, what forms of thinking were students able to use on Type 2 sentences? Were Type 2 sentences able to discriminate between computational and relational thinkers, and among relational thinkers? Were particular forms of explanation –

written descriptions and/or mathematical representations – able to distinguish between students? And finally, were those who showed sound relational thinking on Type 2 sentences able to deal successfully with Type 3 sentences involving literal symbols?

Results and Discussion

All students attempted the addition and subtraction questions involving Type 1, 2 and 3 sentences Some Year 6 students in the Chinese school had difficulty going any further, but this provided sufficient evidence. Year 6 students in the Australian school and Year 7 students in both schools generally completed all, or most of, the questionnaire.

Results on Type 1 Sentences

On Type 1 sentences, relational thinking was evident when, for example, written descriptions, arrows or diagrams were used to compare the size of numbers either side of the equal sign; and where these descriptions, arrows or diagrams were used in chain of argument, based on uncalculated pairs, using compensation and equivalence, to find the value of a missing number. When arrows were used, sometimes they went in the same direction; sometimes in opposite directions on a given item, with the direction of compensation varying accordingly. By contrast, in computational responses, students always completed the calculation on the opposite side to where the number denoted by □ is shown, and then used this result to calculate the value of the missing number.

In solutions to Type 1 sentences, similar, if not always identical, patterns of relational thinking were evident in both countries. Students in the Australian school who used relational thinking generally preferred to use arrows to solve questions involving all four operations. This may have been a reflection of how they have been taught. In the Chinese school, written justifications of relational thinking were more prevalent on questions involving addition and subtraction, but directional arrows were almost universally used by in solving questions involving multiplication and division.

Students were classified as Relational if they successfully gave relational responses on at least one of the four operations. Students were also classified as Relational if they gave a mix of relational and computational responses. Only those students who gave computational responses on all attempted questions were classified as Computational. Even when occasional errors were present among responses, these mistakes had no bearing on the overall classification. Table 1 shows performance across the two schools according to whether students had used computational or relational thinking on Type 1 sentences:

Table 1

Performance on Type 1 Sentences

Schools and numbers at each Year level	Computational		Relational	
Chinese School Year 6 (N = 32)	20	63%	12	37%
Chinese School Year 7 (N = 36)	8	22%	28	78%
Australian School Year 6 (N = 25)	1	4%	24	96%
Australian School Year 7 (N = 71)	17	24%	54	76%

Results on Type 2 and 3 Sentences

Type 2 and Type 3 sentences had been deliberately crafted to "push" students into relational responses even if it was possible for them to complete parts (a) of these questions by computation. Almost all students without exception were able to place numbers correctly in Box A and Box B to make a correct sentence. Some students admittedly chose quite small numbers to place in the boxes to give correct sentences.

Responses to Type 2 sentences

Having constructed several correct sentences in this way, all students attempted to describe the relationship between the numbers in Box A and Box B. However, there was a clear difference between those who simply commented on the numbers used in the boxes, and those whose responses clearly focussed on the conditions needed to make a given sentence correct. For example, in answering Question 2b, some students merely said "Two more" or "Two difference" or "Two between A and B". By contrast, other students used precise relational expressions to answer the same question, such as "Box A needs to be two more than Box B", or wrote "Box B + 2 = Box A". Other clear instances of relational thinking were evident when students used literal symbols in Box A and B such as in Question 4a, $72 - m = 75 - (m + 3)$, or where students used purposefully large numbers, such as for Question 8a, $3 \div 1{,}000{,}000 = 15 \div 5{,}000{,}000$ (Australian Year 7 student); or for question 2a, $18 + 1{,}000{,}000 = 20 + 999{,}998$ (Chinese Year 7 student).

Responses that simply compared the numbers used in Box A and Box B were certainly not wrong, but they appeared to fall short of responses, such as given immediately above. Satisfactory responses to parts a, b and c did not always predict a successful response to part d items which asked "If you put any number in Box A, can you still make a correct sentence?" These part d items distinguished between emergent and clear relational responses. Some students who used emergent relational thinking in parts b and c either rejected the possibility that any number could be used in Box A, or offered incomplete explanations such as "Only if B is correct". Clear relational responses are evident in the following: A Chinese Year 6 student answered Question 2d involving addition by saying, "As long as B is two less than A"; and an Australian Year 7 student answered Question 4d involving subtraction by saying, "Any number can be in Box A so long as Box A is 3 less than Box B, otherwise the result will be disrupted".

Responses to Type 3 sentences

Students who could not give a satisfactory answer to the question, "If you put any number in Box A, can you still make a correct sentence?", could not deal with Type 3 sentences involving *c* and *d* as literal symbols. Many of those whose responses showed emergent relational thinking in parts b and c did not attempt part e items. A few students gave specific values of *c* and *d* that made the accompanying mathematical sentence correct. More frequent instances of incomplete or emergent relational thinking were, "Just put the right numbers in both sides and let the equation be correct", or "*c* is bigger than *d*" (in Questions 2e and 6e) or the reverse (in Questions 4e and 8e). These incomplete responses are contrasted with the following exemplars of relational thinking: "The *d* is worth 8 less than the *c"* (Australian Year 6 student in answer to Question 2e), and "Can! No matter what is A, so long as B = 5 × A (Chinese Year 7 student in answer to Question 8e).

What these exemplars also show is the importance of logical qualifiers in explaining relational thinking. These logical expressions were also evident in responses to Type 1 sentences. However, in the case of Type 2 and 3 sentences, the use of these logical expressions was more varied and prevalent. Mini-arguments, described by Vergnaud (1979), in the form "Because ……therefore….." were consistently used by relational thinkers in their responses to Type 2 and Type 3 sentences, as were logical qualifiers such as "should be", "must be", "has to be", "needs to be" (and their Chinese equivalents). Emergent relational responses rarely included any logical qualifying expressions. Two contrasting responses to item 8e show this. Simply saying that "*d* is bigger than *c*" falls a long way short of saying that "*d* must be three times the value of *c*".

Relational and Emergent Relational Thinkers

Emergent Relational thinkers typically completed the three replicas of the Type 2 sentence by using numbers that were small and/or easy to calculate. Their descriptions of the relationship between the numbers in Box A and Box B in parts b and c generally took the form of a commentary on the numbers used, such as "A is 2 more" [item 2b] instead of a clear statement of what needs to be the case if both sides are to be equivalent. These students did see a pattern between the numbers denoted by Box A and Box B, but whether they understood the general conservation principle that would make such a sentence true was clearly determined by their responses to parts d and e.

In their responses to part d items, Emergent Relational thinkers could not formulate a general statement which would allow *any* number to be used in Box A. Some did say that different values of A and B were possible, or that these values needed to be balanced. They were likely to offer similar incomplete statements in regard to the values of *c* and *d* in part e. Sometimes they gave an incorrect relationship between *c* and *d*, such as $d = 7c$ [item 6e]. Students who gave incomplete relational responses to all attempted items for Type 2 and Type 3 sentences were classified as Emergent Relational. If any mathematically complete explanations were given to these same items, students were classified as Relational. All students could be classified as either Emergent Relational or Relational.

Among Relational thinkers, there was a striking association between making a clear and correct response to part d *and* describing the relationship between the values of *c* and *d* to make the corresponding Type 3 sentence true. Among all Year 6 students, 70% of those who gave a correct response to a part (d) item also correctly described the relationship between *c* and *d* in the corresponding part (e) item. For Year 7 students, in both schools, a successful response to a part d item was followed in 90% of cases by a successful description of the relationship in part e between *c* and *d.* In no case, was a successful response to a part e item preceded by an inadequate response to its related part d

Questions involving two unknowns clearly met their purpose of pushing students beyond computation. In addition, there were students in the Australian school and in the Chinese school whose responses were consistently and clearly relational in regard to *all* attempted questions relating Type 2 and 3 sentences. These students' responses are referred to in the notes to Table 2 which shows the relationship between computational and relationship responses to Type 1 sentences and responses to Types 2 and 3 sentences.

Table 2

Comparing Performances on Type 1 with Performances on Types 2 and 3

	Type 1	Types 2 & 3	
Chinese	Year 6 Computational (n =20)	15	5
	Year 6 Relational (n = 12)	4	8
	Year 7 Computational (n = 8)	5	3
	Year 7 Relational (n =28)	0	28[a]
Australian	Year 6 Computational (n = 1)	1	0
	Year 6 Relational (n = 24)	11	13[b]
	Year 7 Computational (n = 17)	13	4
	Year 7 Relational (n =54)	0	54[c]
		Emergent	**Relational**

Note a: 6 out these 28 students successfully completed all questions; **Note b**: 3 out these 13 students successfully completed all questions; **Note c**: 28 out these 54 students successfully completed all questions

Analysing the performances of the four different groups does allow several important conclusions to be drawn concerning some key junctures in relational thinking.

Conclusions

From Table 2, it can be seen that fifteen Chinese Year 6 students used computational approaches on Type 1 sentences *and* were unable to deal successfully with Type 2 or 3 sentences. Their difficulties with Type 2 and Type 3 sentences suggest strongly that these students are restricted to using computational methods. Five students who chose to work computationally on Type 1 sentences were able to shift into relational thinking when confronted with sentences involving two unknowns. By contrast, eight students out of the twelve who used relational approaches to deal with Type 1 sentences were able to deal successfully with Type 2 or Type 3 sentences, even if they did not cover all questions.

The Year 6 Australian group showed clear evidence of relational thinking on Type 1 sentences, even if some students were not able to complete all questions on the questionnaire. This may well be attributed to the explicit emphasis that their teacher places on relational methods to solve Type 1 sentences. This Australian Year 6 group demonstrates very clearly that successful use of relational approaches to solve sentences involving one unknown does not translate automatically into success in dealing with sentences involving two unknowns. Almost half of those who had used relational approaches Type 1 sentences had some difficulty with Type 2 sentences. Transition across this juncture can by no means be assumed, possibly because the second type of sentence is structurally more complex and mathematically more demanding for students. This also suggests that specific attention needs to be given to Type 2 sentences.

By contrast, the majority of students in Year 7 in both the Chinese and Australian school employed relational approaches to deal with Type 1 sentences, *and* were generally capable of dealing successfully with Type 2 and 3 sentences. The juncture between relational thinking on numbers sentences involving one missing number and those involving two unknown numbers appears more assured with these Year 7 students. This is not to suggest that relational thinking can be assumed among Year 7 students. Previous studies show the presence of relational thinking is subject to wide variation between schools. The careful use logically qualifying expressions by these Year 7 students, and by some students in Year 6, is a further area ripe for investigation.

A relatively small number of students in both schools used computational approaches on Type 1 sentences and were able to shift gear, so to speak, when they needed to think relationally on Type 2 and Type 3 sentences. It is, however, sobering to note that among Year 7 students in both the Chinese and the Australian school there are still students who seem to be restricted to using only computation approaches. Thirteen of the seventeen computational thinkers in the Australian school fell into this category as did five out of eight computational thinkers in the Chinese school. It can be anticipated that these Year 7 students are likely to experience serious difficulties in the learning of algebra.

The strikingly close association between successfully identifying conditions under which "any number could be used in Box A" and a successful response to the corresponding Type 3 sentence points to a key juncture between relational thinking on Type 2 number sentences and an ability to explain relationships between literal symbols. Greater attention should be given to using Type 2 number sentences as a bridge to algebra.

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